Fourier 1-norm and quantum speed-up

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Overview

Decision trees

- We want to compute a Boolean function.
- The depth of the tree represents the complexity.
- A randomized tree is a probabilistic distribution over deterministic trees.

The quantum query model I

- The states of our computer are described by unit vectors in a Hilbert space $\mathcal{H}$, whose basis is $|i\rangle |j\rangle$, where $i \in \{0, 1, .., n\}$ and $j \in \{1, .., m\}$.
- We have a set of unitary operators $\{U_i\}$ over $\mathcal{H}$.
- We denote a query operator O_x, such that $O_x |i\rangle |j\rangle = (-1)^{x_i} |i\rangle |j\rangle$, where $x \equiv x_0 x_1 \cdots x_n$ is the input, and $x_0 \equiv 0$.

The quantum query model II

- The initial state of the algorithm is $|0\rangle |0\rangle$.
- The final state of the algorithm over input x is defined as $\left|\Psi_x^f\right\rangle = U_t O_x U_{t-1} ... O_x U_0 |0\rangle |0\rangle$.
- We denote a query operator O_x, such that $O_x |i\rangle |j\rangle = (-1)^{x_i} |i\rangle |j\rangle$, where $x \equiv x_0 x_1 \cdots x_n$ is the input, and $x_0 \equiv 0$.

The quantum query model III

CSOP

An indexed set of pairwise orthogonal projectors $\{P_z : z \in T\}$ is called a Complete Set of Orthogonal Projectors if it satisfies

$$\sum_{z \in T} P_z = I_{\mathcal{H}}. \tag{1}$$

- The probability of obtaining the output $z \in T$ is $\pi_z(x) = \left\| P_z \left| \Psi_x^f \right\rangle \right\|^2$.
- An algorithm computes a function $f : D \to T$ within error ε if $\pi_{f(x)}(x) \geq 1 - \varepsilon$ for all input $x \in D \subset \{0,1\}^n$.

The Fourier basis I

We consider the Fourier basis for the vector space of all functions $f : \{0,1\}^n \to \mathbb{R}$ given by the functions

$$\chi_b : \{0,1\}^n \to \{1,-1\},$$

such that $\chi_b(x) = (-1)^{b \cdot x}$ for $b \in \{0,1\}^n$ and $b \cdot x = \sum_i b_i x_i$. This family contains a constant function that we denote as $\chi_0 = 1$.

The Fourier basis II

Any function $f : \{0,1\}^n \to \mathbb{R}$ has a unique representation as a linear combination

$$f = \sum_{b \in \{0,1\}^n} \alpha_b \chi_b. \qquad (2)$$

Metrics

1-norm

we denote the Fourier 1-norm of f as

$$L(f) = \sum_{b \in \{0,1\}^n} |\alpha_b| . \quad (3)$$

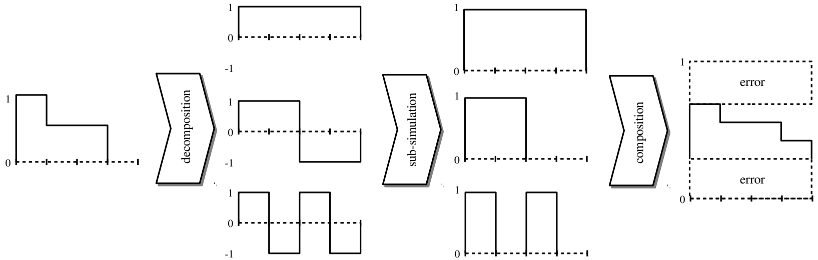
Degree

Another measure is the degree of f, which is defined as

$$\deg(f) = \max_{|b|} \{b : \alpha_b \neq 0\} . \quad (4)$$

The intuition

- Any output probability can be decomposed in functions χ_b.
- We define a classical simulation for each χ_b.



Notation

- $R_\varepsilon(f)$ denotes the minimum number of queries that are necessary for computing f within error ε by a randomized decision tree.
- $\pi_1(x)$ is the probability of a quantum algorithm returning output 1 for a given input x.

First Bound

Theorem

Consider $D \subset \{0,1\}^n$ and a function $f : D \to \{0,1\}$ that is computed within error $\varepsilon > 0$ and t queries, by a quantum query algorithm. If we define

$$F_\varepsilon(l) = \left\lceil \frac{-16 \ln(\varepsilon)(1+l)(1+l-\varepsilon)}{(1-2\varepsilon)^2} \right\rceil, \tag{5}$$

then

$$\frac{R_\varepsilon(f)}{t} \leq F_\varepsilon(L(\pi_1)). \tag{6}$$

Second Bound

Theorem

Consider $D \subset \{0,1\}^n$ *and a function* $f : D \to \{0,1\}$ *that is* ε*-approximated by a polynomial* $p : \mathbb{R}^n \to \mathbb{R}$*. If* $\deg(p) \leq 2t$, *then*

$$\frac{R_\varepsilon(f)}{2t} \leq F_\varepsilon(L(p)). \tag{7}$$

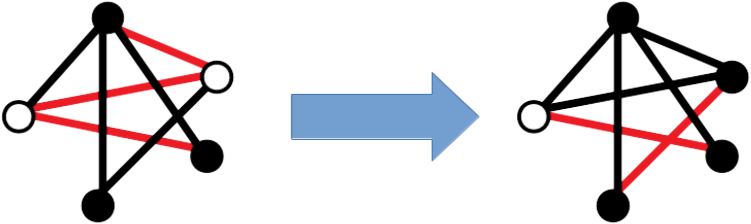
A characterization between degree and query complexity.

A partial Boolean function f is computable by a 1-query quantum algorithm with error bounded by $\epsilon < 1/2$ iff f can be approximated by a degree-2 polynomial with error bounded by $\epsilon' < 1/2$.

Fourier analysis of degree-2 polynomials.

- The Fourier spectrum is composed by Walsh functions of the form $\chi(x) = -1^{\left(ax_i + bx_j\right)}$.
- Each Walsh function is affected by at most two values of the input x.

1-query algorithms as weighted dynamic graphs



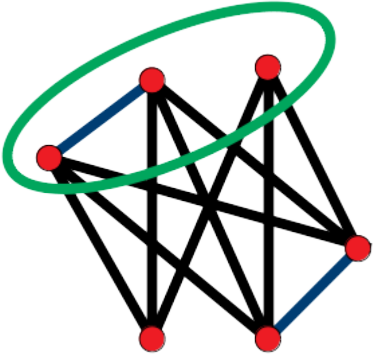
Graphs that maximize L-1 norm

How to maximize L-1 norm on graph based quantum query algorithms

- Low difference between weights.
- Minimizing the upper-bound value of the sum weight for every vertex and configuration.

An special case

Consider a regular graph with the following property: The edges of the graph intersect almost half of the edges of any complete bipartite graph that has the same vertices.



Conclusions

- Weighted graphs are an alternative representation for 1-query quantum algorithms.
- Such representation gives a direct upper bound for quantum speed up over a similar classical algorithm.

The End